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A Characterization of Hering's Plane of Order 27

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Abstract. Hering's translation plane of order 27 has been characterized by its order and the fact that it admits SL(2,13) in its translation complement (see [1]). We show that, aside from the Desarguian plane and a Generalized André plane, it is the only plane of order 27 which admits a subgroup of SL(2,13) of order 13×12.

1. Introduction

In section 2 we review the relevant subgroups of SL(2,13), GL(3,3) and GL(6,3). We show in proposition 1 that GL(6.3) contains a unique conjugacy class of subgroups of order 13×12 isomorphic to a subgroup of SL(2,13) of the same order.

In section 3 we introduce three spread sets which admit a given group of order 13×12. We prove in proposition 2 that there are no more such spread sets. The next lemma characterizes the planes arising from these spread sets and the following theorem is deduced:

Theorem. *The only translation planes of order* 27 *which admit a subgroup of order* 13×12, *isomorphic to that of* SL(2,13), *in their translation complement are the Desarguian plane, a Generalized André plane and Hering's translation plane of order* 27.

Our notation is standard and we follow [2].

2. Subgroups of SL(2,13), GL(3,3) and GL(6,3)

Since the order of the normalizer of a Sylow 13-subgroup of SL(2,13) is 13×12, it follows that SL(2,13) has a unique conjugacy class of subgroups of order

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13×12. One representative of this class is generated by the concrete matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}.$$

As an abstract group it can be presented as

$$S = \langle h, c \mid h^{13} = c^{12} = 1, c^{-1}hc = h^{10} \rangle.$$

The Sylow 13-subgroup of $\mathrm{GL}(3,3)$ is of order 13. From the factorization

$$\begin{aligned} X^{13} - 1 =& (X-1)(X^3-1)(X^3-X-1)(X^3+X^2+X-1) \\ & (X^3+X^2-1)(X^3-X^2-X-1) \end{aligned}$$

in irreducible factors over $\mathrm{GF}(3)$, we see that there are four conjugacy classes of elements of order 13 in $\mathrm{GL}(3,3)$. We can choose h with minimal polynomial $X^3 - X - 1$ to generate the Sylow 13-subgroup of $\mathrm{GL}(3,3)$. In rational canonical form

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and if

$$x = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

then $|x| = 3$ and $x^{-1}hx = h^3$. We have that

$$N_{\mathrm{GL}(3,3)}(\langle h \rangle) = [\langle -I \rangle \times \langle h \rangle] \rtimes \langle x \rangle.$$

The conjugacy classes of elements of order 13 are represented in $\langle h \rangle$ by h, h^2, h^4, h^8.

A Sylow 13-subgroup of $\mathrm{GL}(6,3)$ is of order 13^2 and,

$$T = \left\{ \begin{pmatrix} h^i & 0 \\ 0 & h^j \end{pmatrix} \mid i, j = 0, \ldots, 12 \right\}$$

being one them, it follows that every element of order 13 of $\mathrm{GL}(6,3)$ is conjugate to an element of the form $\begin{pmatrix} h^i & 0 \\ 0 & j^j \end{pmatrix}$. It can be deduced that $\mathrm{GL}(6,3)$ also has four conjugacy classes of elements of order 13 which are represented in T by

$$\begin{pmatrix} I & 0 \\ 0 & h \end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & h^2 \end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & h^4 \end{pmatrix}.$$

If we let $C = \begin{pmatrix} 0 & -I \\ x^2 & 0 \end{pmatrix}$ then C is of order 12. Moreover if $H = \begin{pmatrix} h & 0 \\ 0 & h^4 \end{pmatrix}$ then $C^{-1}HC = H^{10}$ and the group $G = \langle H, C \rangle$ is a subgroup of $\mathrm{GL}(6,3)$ isomorphic to S.

Proposition 1. *If G_0 is a subgroup of* $\mathrm{GL}(6,3)$ *and G_0 is isomorphic to S, then G_0 is conjugate to G in* $\mathrm{GL}(6,3)$.

Proof. We may assume by Sylow's theorem that G_0 contains one of

$$\begin{pmatrix} I & 0 \\ 0 & h \end{pmatrix}, \quad \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad \begin{pmatrix} h & 0 \\ 0 & h^2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} h & 0 \\ 0 & h^4 \end{pmatrix}.$$

To see that the first three options cannot occur, we need only remark that if $ah^i = h^j a$ with any 3×3 matrix a over $\mathrm{GF}(3)$ then $i = j = 0$ or $a = 0$ or a is invertible. In the latter case h^i is conjugate to h^j. If we had, for instance, $\begin{pmatrix} I & 0 \\ 0 & h \end{pmatrix} \in G_0$ and if $C_0 = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ was such that

$$C_0^{-1} \begin{pmatrix} I & 0 \\ 0 & h \end{pmatrix} C_0 = \begin{pmatrix} I & 0 \\ 0 & h \end{pmatrix}^{10}$$

then $Y(h^{10} - I) = (h - I)Z = 0$ and $hW = Wh^{10}$ would lead to $Y = Z = W = 0$, a contradiction. (Note that $-h$ generates a subgroup of $\mathrm{GL}(6,3)$ which is the multiplicative group of $\mathrm{GL}(27)$ with the addition of matrices.) The same type of argument eliminates the other two and we may assume that $H \in G_0$.

Let $C_0 \in G_0$ be such that $C_0^{-1}HC_0 = H^{10}$, say $C_0 = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$. Then we get $X = W = 0$ and Y, $Zx^{-2} \in C_{\mathrm{GL}(3,3)}(h) = \langle -I \rangle \times \langle h \rangle$. This shows that there are 4×13^2 possibilities for C_0 in $\mathrm{GL}(6,3)$. Since

$$\begin{pmatrix} 0 & x \\ -I & 0 \end{pmatrix} \begin{pmatrix} h^i & 0 \\ 0 & h^j \end{pmatrix} \begin{pmatrix} 0 & -I \\ x^2 & 0 \end{pmatrix} = \begin{pmatrix} h^{9j} & 0 \\ 0 & h^i \end{pmatrix},$$

we see that C normalizes T, but C fixes no element of T. Therefore in the group $F = T \rtimes \langle C \rangle$, since T does not normalize $\langle C \rangle$ and no element of T centralizes $\langle C \rangle$, we have that $N_F(\langle C \rangle) = \langle C \rangle$. Thus $\langle C \rangle$ has 13^2 conjugates in F. Since both C and $-C$ conjugate H to H^{10}, this accounts for 2×13^2 of the possibilities and the proof would be finished if C_0 was one of these, as the conjugation of C to C_0 occurs within $F \subseteq N(\langle H \rangle)$. But this is indeed the case as multiplication by $J = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ leaves invariant the set of all 4×13^2 matrices $\begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix}$ with Y, $Zx^{-2} \in \langle -I \rangle \times \langle h \rangle$ and, if $\tilde{C}$ is conjugate to C then $\tilde{C}^6 = -I$ while

$(J\tilde{C})^6 = I$. Thus the remaining 2×13^2 are not matrices of order 12, but are of order 6. This completes the proof.

3. Spreads admitting the group G

It is easy to see that $C_{\mathrm{GL}(3,3)}(x) = \langle -I \rangle \times \langle x \rangle \times \langle w \rangle$ where w is an element of order 3 with minimal polynomial $(X-1)^2$. If x is chosen as in 2 then we can put

$$w = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

Proposition 2. *Let V denote a 3-dimensional vector space over* $\mathrm{GF}(3)$. *Then the following are the only three non-isomorphic spread sets on $V \oplus V$ which define planes admitting the group $G = \langle H, C \rangle$.*

1) $\Sigma_1 = \langle h \rangle x^2 \cup \langle h \rangle (-x^2)$

2) $\Sigma_2 = \langle h \rangle \cup \langle h \rangle (-x)$

3) $\Sigma_3 = \{(-1)^j h^i w^j x^2 h^{9i} \mid i = 0, \ldots, 12; j = 1, 2\}$

Proof. Let $\Sigma \subseteq \mathrm{GL}(V)$ be an arbitrary spread set defining a spread on $V \oplus V$ which admits the group G. By [2] the spread is of the form $\{(V,0),(0,V)\} \cup \{(V, VM) | M \in \Sigma\}$. Since H fixes both $(V,0)$ and $(0,V)$ while C interchanges them, it follows that $G = \langle H, C \rangle$ permutes $\{(V, VM) | M \in \Sigma\}$. Since H has no other proper invariant subspaces than $(V,0)$ and $(0,V)$, it follows that H has two orbits of length 13 on $\{(V, VM) | M \in \Sigma\}$. Thus:

$$\{(V, VM) \mid M \in \Sigma\} = \{(V, VM)H^i \mid i = 0, \ldots, 12\} \cup \\ \cup \{(V, VM')H^i \mid i = 0, \ldots, 12\}$$

where M, M' are chosen so that (V, VM) and (V, VM') are in the distinct orbits of H.

Since C normalizes $\langle H \rangle$ so does $C^4 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, which is of order 3. Thus C^4 permutes the orbits of $\langle H \rangle$ and, since it fixes $(V,0)$ and $(0,V)$, it must also fix the two other orbits. But then, being of order 3, it must fix at least one component in each. We may assume that these components are the $(V, VM), (V, VM')$ chosen above. On a fixed component C^4 fixes the origin and the point at infinity, therefore it fixes at least two of the remaining 26 points. It follows that C^4 is planar and, since $6^2 + 6 > 27$, the order of the fixed subplane must be 3 and C^4 has no other fixed component than $(V,0),(0,V),(V,VM),(V,VM')$. We conclude also that $M = x^{-1}Mx, M' = x^{-1}M'x$ and so $M,\ M' \in C_{\mathrm{GL}(3,3)}(x)$.

Now C^3 is of order 4, it commutes with C^4, thus it permutes the fixed elements of C^4. Since it interchanges $(V,0)$ and $(0,V)$, it must fix both (V,VM), (V,VM') or interchanges them. From $(V,VM)C^3 = (V,VM)$ it follows that $M = -M^{-1}x$ which is a contradiction as $M^2 = -x$ would imply that M is an element of order 12 in $C_{\mathrm{GL}(3,3)}(x)$. Since there is no such elements in $C_{\mathrm{GL}(3,3)}(x)$ we conclude that $M' = -M^{-1}x$.

We have shown that the spread

$$\{(V,VM)H^i \mid i = 0,\dots,12\} \cup \{(V,V(-M^{-1}x))H^i \mid i = 0,\dots,12\}$$

is determined by $M \in C_{\mathrm{GL}(3,3)}(x)$.

It is an easy calculation to see that for $M = \pm w,\ \pm w^2,\ \pm wx$ or $\pm w^2x$ we do not get a spread.

The possible spreads are then determined by the pairs $\{M, -M^{-1}x\} = \{-I,x\},\ \{I,-x\},\ \{x^2,-x^2\},\ \{wx^2,-w^2x^2\}$ or $\{w^2x^2,-wx^2\}$.

The first two pairs produce isomorphic spreads, as can be seen using [2; 26], as the spread set given by $\{-I,x\}$ becomes that given by $\{I,x\}$ under the action of $\sigma = \begin{pmatrix} h & 0 \\ 0 & -I \end{pmatrix}$. By the same result, applying $\lambda = \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}$ to the spread defined by $\{wx^2, -w^2x^2\}$ and applying $\mu = \begin{pmatrix} w & 0 \\ 0 & -z \end{pmatrix}$ to that given by $\{w^2x^2, -wx^2\}$, the spreads coincide and the last two pairs produce isomorphic planes.

The spread Σ_1 is defined by $\{x^2,-x^2\}$, Σ_2 is defined by $\{I,-x\}$ and Σ_3 is defined by $\{w^2x^2,-wx^2\}$. The proof of the proposition will be completed once we establish that these are non-isomorphic. Both this and the proof of the theorem follow from the next lemma.

Lemma 3. *If Π_i denotes the plane defined by the spread set Σ_i then Π_1 is Desarguian, Π_2 is a generalized André plane and Π_3 is Hering's plane of order 27.*

Proof. Since

$$(V, Vh^i x^2) \begin{pmatrix} h & 0 \\ 0 & h^9 \end{pmatrix} = (V, Vh^i x^2)$$

and

$$(V, Vh^i(-x^2)) \begin{pmatrix} h & 0 \\ 0 & h^9 \end{pmatrix} = (V, Vh^i(-x^2)),$$

$(i = 0, \ldots, 12)$, it follows that the kernel of the plane defined by Σ_1 contains a group isomorphic to $\langle h \rangle$. Since it also contains $-I$ and since $\langle h \rangle \times \langle -I \rangle \cup \{0\}$ is isomorphic to $\mathrm{GF}(27)$ as a field, it follows that Π_1 is Desarguian [2].

The plane defined by Σ_2 does not admit $(x, y) \to (y, x)$ as $(V, V(-h^j x)) \to (V, V(-h^k x^2))$ and $-h^k x^2 \notin \Sigma_2$. If

$$\sigma = \begin{pmatrix} I & 0 \\ 0 & h \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix}$$

then a straightforward calculation shows that σ is an $((\infty), (V, 0))$-homology of order 13, τ is a $((0), (0, V))$-homology of order 13 and each component is in an orbit of length at most 13 under $\langle \sigma \rangle \times \langle \tau \rangle$. Since 13 is a primitive divisor of $3^3 - 1$, the plane defined by Σ_2 is a generalized André plane by [3; 5.2.1].

Finally, the spread defined by Σ_3 is invariant under

$$W = -\begin{pmatrix} xwu & x^2 w^2 u \\ x^2 w^2 u & -wu \end{pmatrix}$$

where u is an involution inverting x and commuting with w. If x, w are chosen as before, one can choose

$$u = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

But then since $\langle W, H, C \rangle \cong \mathrm{SL}(2, 13)$ as can be seen in [1], it follows from the same reference that the plane defined by Σ_3 is Hering's plane of order 27. This completes the proof of the lemma, the proposition and the theorem.

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